Squared Choices

Submission deadline: May 28^{th} 2023

Denote $\dfrac{n!}{k!(n-k)!}$ by $\dbinom{n}{k}$ where $k \leq n$ and both are natural numbers.
Evaluate

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2$$

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Discussion:

Since

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

and

$$(1+y)^n = \binom{n}{0} + \binom{n}{1}y + \binom{n}{2}y^2 + \cdots + \binom{n}{n}y^n$$

it is easy to see that

$$(1+x)^n(1+y)^n = \binom{n}{0}^2 + xy\binom{n}{1}^2 + \cdots + x^n y^n \binom{n}{n}^2 + \sum_{p\neq q} x^p y^q \binom{n}{p}\binom{n}{q}$$

Let $y = 1/x$ to get

$$(1+x)^n \frac{(1+x)^n}{x^n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 + \sum_{p\neq q} x^{p-q} \binom{n}{p}\binom{n}{q}$$

Thus,

$$(1+x)^{2n} = x^n \left[\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2\right] + \sum_{p\neq q} x^{n+p-q} \binom{n}{p}\binom{n}{q}$$

Now, by comparing the coefficients of x^n in each side of the equation above it follows that

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$