Integral Roots

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Suppose that the polynomial

$$P(x) = a_0x^{2023} + a_1x^{2022} + \cdots + a_{2022}x + a_{2023}$$

has integral coefficients. If $P(0)$ and $P(1)$ are odd values, does the equation $P(x) = 0$, have integral roots?

The problem was solved by

- Elisaveta Andrushkevich, *Southern Federal University, Russia.*

- Muhammed Yuksel, *Ankara, Turkey.*

- Atakan ERDEM, *Middle East Technical University, Ankara, Turkey.*

- Seth Cohen, *Concord, New Hampshire, USA.*

Discussion

Since $P(0) = a_{2023}$, it easily follows that a_{2023} is odd. Moreover, since $P(1) = a_0 + a_1 + \cdots + a_{2022} + a_{2023}$, and a_{2023} are odd, it is easy to see that $a_0 + a_1 + \cdots + a_{2022}$ is even.

Let k be any integer.

If i is a natural number, then $(2k+1)^i$ is odd. Thus, let $2s_i + 1 = (2k+1)^i$. Then,

$$P(2k+1) = 2(a_0 s_0 + \cdots + a_{2022} s_{2022}) + (a_0 + \cdots + a_{2022}) + a_{2023}$$

Since both $2(a_0 s_0 + \cdots + a_{2022} s_{2022}), (a_0 + \cdots + a_{2022})$ are even and a_{2023} is odd, it follows that $P(2k+1) \neq 0$.

Now,

$$P(2k) = a_0 2^{2023} k^{2023} + \cdots + a_{2022} 2k + a_{2023}.$$

Since $a_0 2^{2023} k^{2023} + \cdots + a_{2022} 2k$ is even and a_{2023} is odd, it follows that $P(2k) \neq 0$.

Thus, $P(x) = 0$ has no integral roots.